Linear Algebra and its Applications 613 (2021) 151–169

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Diagonalizable higher degree forms and symmetric tensors ☆

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A R T I C L E I N F O

Article history:
Received 6 April 2020
Accepted 16 December 2020
Available online 29 December 2020
Submitted by A. Uschmajew

MSC:
15A69
14J70
11E76

Keywords:
Higher degree forms
Diagonalization
Symmetric tensor

A B S T R A C T

We provide simple criteria and algorithms for expressing homogeneous polynomials as sums of powers of independent linear forms, or equivalently, for decomposing symmetric tensors into sums of rank-1 symmetric tensors of linearly independent vectors. The criteria rely on two facets of higher degree forms, namely Harrison's algebraic theory and some algebro-geometric properties. The proposed algorithms are based purely on solving linear and quadratic equations. Moreover, as a byproduct of our criteria and algorithms one can easily decide whether or not a homogeneous polynomial or symmetric tensor is orthogonally or unitarily decomposable.

1. Introduction

The paper is concerned about the problem: given a homogeneous polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ of degree $d \geq 3$, decide whether it is equivalent to $x_1^d + \cdots + x_n^d$

☆ Supported by NSFC 11971181 and 11971449.
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https://doi.org/10.1016/j.laa.2020.12.018
0024-3795/

(called diagonalizable) and if so, give efficient algorithms to express the equivalence. It has an equivalent version in terms of symmetric tensors: given a symmetric d-tensor $A = (a_{i_1 \cdots i_d})_{1 \leq i_1, \cdots, i_d \leq n}$, provide criteria and algorithms to express $A = v_1^{\otimes d} + \cdots + v_n^{\otimes d}$ where the v_i's are a basis of $\mathbb{C}^n$. Though this is a typical problem of classical invariant theory (see e.g. [9]), especially in the polynomial version, it has aroused much interest in communities of applied mathematics and computing sciences. As a matter of fact, the problem has been explicitly proposed in some previous papers [4,11,14] of signal processing and computational complexity. In addition, assuming a homogeneous polynomial is equivalent to a sum of powers of independent linear forms, people are interested in whether one can choose the change of variables to be orthogonal or unitary. See for example [15,19,1,13] among many works in this direction.

In this paper we tackle the problem (mainly in the polynomial version) through two important aspects of higher degree forms (another popular synonym of homogeneous polynomials of degree at least 3): the algebraic theory, a higher analogue of Witt's algebraic theory of quadratic forms, initiated by Harrison [6,7], and algebro-geometric properties of sums of powers. The crux is Harrison's centers of higher degree forms, which can be seen as a generalization of symmetric matrices. In particular, the direct sum decompositions of a form are in bijection with the decompositions of the unit of its center algebra into orthogonal idempotents. On the other hand, it is obvious that diagonalizable forms enjoy nice algebro-geometric properties. Specifically, they are smooth and this imposes very strong restriction, namely the semisimplicity, on their centers. With this one can easily show that a form $f \in \mathbb{C}[x_1, \cdots, x_n]$ is diagonalizable if and only if its center algebra is $Z(f) \cong \mathbb{C} \times \cdots \times \mathbb{C}$ (n copies). This enables us to provide several criteria and algorithms for diagonalizing higher degree forms or symmetric tensors. We present in detail a simple algorithm involving only linear equations (for computing the center) and quadratic equations (for computing orthogonal primitive elements and presenting the explicit diagonalization). Furthermore, with the help of Harrison's uniqueness of the decomposition of a form into a direct sum of indecomposable forms [6], we notice that the orthogonal or unitary decomposability of a form (over appropriate ground fields) is a property of its diagonalizability. Once the diagonalization is determined, one can decide whether a form is orthogonally or unitarily decomposable by a straightforward check on any chosen change of variables for the diagonalization. As an example to illuminate our approach, we provide a simple proof for a main result of [1,13].

The theory of Harrison's centers was also used to study polynomial equivalence in the literature. In his thesis [20], Saxena introduced a class of cubic forms which can capture the isomorphism problem of commutative algebras and he obtained the indecomposability of the cubic forms via Harrison's theory. Direct sum decompositions of higher degree forms have also been approached via the Apolarity Lemma, see e.g. [12,2]. In particular, the centers of higher degree forms were rediscovered in [12]. However a full application to direct sum decompositions via centers was not pursued therein. Numerical approaches were also applied to orthogonal decompositions of symmetric tensors. For example, in [16] Kolda showed that it is possible to solve the symmetric orthogonal tensor decomposition algorithm via a straightforward matrix eigenproblem with the assumption that the orthogonal decomposition exists or with a small amount of noise. This is compatible with our findings. In particular, based on the criteria of diagonalizability (see Theorem 4.4) we find that Kolda's result can be extended to all diagonalization of symmetric tensors, see Remark 4.7.

It is natural to extend the main strategy of the present paper to two more general problems. One is the Waring decompositions of forms, that is decomposing forms into sums of powers of linear forms which are linearly dependent in general. In terms of symmetric tensors, this is the decompositions into symmetric rank-1 terms. A possible way is to perturb a given nondiagonalizable form into a diagonalizable one in more variables. The other is the direct sum decompositions of forms, or equivalently the decompositions of symmetric tensors into sums of block terms. This is reduced to a thorough understanding of the semisimple quotient of the center algebra of any given form. These problems will be addressed in our forthcoming work.

The remainder of this paper is organized as follows. In Section 2 we recall the basic notions and mutual interpretations of higher degree forms, symmetric tensors and symmetric multilinear spaces. Section 3 is devoted to centers and direct sum decompositions of forms. The main results, criteria and algorithms for (orthogonal or unitary) diagonalizations of higher degree forms, are presented in Section 4. In Section 5 we provide some examples to elucidate the criteria and algorithms. Throughout the paper, let $d \geq 3$ be an integer and we consider forms and symmetric tensors of degree d. Although our motivating problem is over the complex numbers $\mathbb{C}$, in most cases we can work over a general field $\Bbbk$ with $\operatorname{char} \Bbbk = 0$, or $\operatorname{char} \Bbbk > d$.

2. Higher degree forms, symmetric tensors and symmetric multilinear spaces

Higher degree forms are homogeneous polynomials of degree $d \geq 3$. Similar to the familiar situation of quadratic forms, higher degree forms are naturally associated to symmetric tensors (i.e., symmetric multi-dimensional matrices) and to symmetric multilinear spaces.

Let $f(x_1, \cdots, x_n) \in \Bbbk[x_1, \cdots, x_n]$ be a form of degree d. For convenience, we write f in the symmetric way:

$$f(x_1, \cdots, x_n) = \sum_{1 \leq i_1, \cdots, i_d \leq n} a_{i_1 \cdots i_d} x_{i_1} \cdots x_{i_d}$$

where the $a_{i_1 \cdots i_d}$'s are symmetric with respect to their indices. The resulting symmetric d-tensor $A = (a_{i_1 \cdots i_d})_{1 \leq i_1, \cdots, i_d \leq n}$ is called the associated symmetric tensor of f. We also write the form $f(x_1, \cdots, x_n) = Ax^d$ in terms of products of tensors (see e.g. [21]), where $x = (x_1, \cdots, x_n)^T$ is the vector of variables. Corresponding to the form f there is also an associated symmetric d-linear space. Let V be a vector space over $\Bbbk$ of dimension n with a basis $\alpha_1, \cdots, \alpha_n$. Define $\Theta \colon V \times \cdots \times V \longrightarrow \Bbbk$ by

$$\Theta(\alpha_{i_1}, \cdots, \alpha_{i_d}) = a_{i_1 \cdots i_d}, \quad \forall 1 \leq i_1, \cdots, i_d \leq n.$$

The pair (V, Θ) is called the associated symmetric d-linear space of f under the basis $\alpha_1, \cdots, \alpha_n$. One can recover the form f from (V, Θ) as

$$f(x_1, \cdots, x_n) = \Theta\left(\sum_{1 \leq i \leq n} x_i \alpha_i, \ \ldots, \ \sum_{1 \leq i \leq n} x_i \alpha_i\right).$$

If $x = Py$ with $P = (p_{ij}) \in \mathrm{GL}(n, \Bbbk)$ is an invertible change of variables, then the resulting form is

$$g(y_1, \cdots, y_n) = \sum_{1 \leq j_1, \cdots, j_d \leq n} \ \sum_{1 \leq i_1, \cdots, i_d \leq n} a_{i_1 \cdots i_d} p_{i_1 j_1} \cdots p_{i_d j_d} y_{j_1} \cdots y_{j_d}$$

and the associated symmetric tensor becomes

$$AP^d := \left(\sum_{1 \leq i_1, \cdots, i_d \leq n} a_{i_1 \cdots i_d} p_{i_1 j_1} \cdots p_{i_d j_d}\right)_{1 \leq j_1, \cdots, j_d \leq n}.$$

We call AP^d the d-congruence of A by P. Let $(\beta_1, \cdots, \beta_n) = (\alpha_1, \cdots, \alpha_n)P$, then under this new basis the associated symmetric d-linear space reads

$$\Theta(\beta_{j_1}, \cdots, \beta_{j_d}) = \sum_{1 \leq i_1, \cdots, i_d \leq n} a_{i_1 \cdots i_d} p_{i_1 j_1} \cdots p_{i_d j_d}, \quad \forall 1 \leq j_1, \cdots, j_d \leq n.$$

The form f is called diagonalizable over $\Bbbk$ if

$$f(x_1, \cdots, x_n) = \sum_{1 \leq i \leq r} \lambda_i l_i(x_1, \cdots, x_n)^d,$$

where the λ_i's are nonzero constants in $\Bbbk$ and the l_i's are independent $\Bbbk$-linear forms. Clearly, if this is the case then $r \leq n$. Accordingly, the symmetric tensor A is called diagonalizable over $\Bbbk$ if there exists a $P \in \mathrm{GL}(n, \Bbbk)$ such that AP^d is diagonal (i.e., the entries are 0 unless the indices are identical), and the symmetric d-linear space (V, Θ) is called diagonalizable if there exists a basis $\beta_1, \cdots, \beta_n$ of V such that $\Theta(\beta_{j_1}, \cdots, \beta_{j_d}) = 0$ unless $j_1 = \cdots = j_d$. Moreover, if $l_i(x_1, \cdots, x_n) = v_{i1}x_1 + \cdots + v_{in}x_n$ $(1 \leq i \leq r)$ and denote $v_i = (v_{i1}, \cdots, v_{in})$, then the corresponding decomposition of the associated symmetric tensor is

$$A = \lambda_1 v_1^{\otimes d} + \cdots + \lambda_r v_r^{\otimes d}.$$

A diagonalizable form is called orthogonally or unitarily diagonalizable if there is an orthogonal or unitary change of variables for the diagonalization. Of course, in this case one needs to work on appropriate ground fields so that orthogonal and unitary groups are well-defined. We also remark that one may replace r by n in the previous definition, as one can consider only nondegenerate forms without loss of generality. See Section 3 for more explicit explanation.

In the following we don't distinguish the three synonyms, namely higher degree forms, symmetric tensors and symmetric multilinear spaces. The discussions are mostly presented in terms of higher degree forms. One can easily shift to the versions of symmetric tensors and symmetric multilinear spaces.

3. Centers and direct sum decompositions

In his pioneering work [6] of algebraic theory of higher degree forms, Harrison introduced the notion of centers (in terms of symmetric multilinear spaces) to deal with the direct sum decompositions of forms.

Definition 3.1. Let (V, Θ) be a symmetric d-linear space. The center, denoted by $Z(V, \Theta)$, of (V, Θ) is defined as

$$\{\phi \in \mathrm{End}(V) \mid \Theta(\phi(v_1), v_2, \cdots, v_d) = \Theta(v_1, \phi(v_2), \cdots, v_d),\ \forall v_1, v_2, \cdots, v_d \in V\}.$$

Let f be the associated degree d form of (V, Θ) under a basis $\alpha_1, \cdots, \alpha_n$. By H we denote the Hessian matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j})_{1\le i,\ j\le n}$ of the form f and by $A^{(i_3\cdots i_d)}$ the $n \times n$ matrix $(a_{i_1 i_2 i_3\cdots i_d})_{1\le i_1,i_2\le n}$ where $A = (a_{i_1\cdots i_d})_{1\le i_1,\cdots,i_d\le n}$ is the associated symmetric d-tensor of f. Then we have the following equivalent definitions of centers in terms of forms (see [7]) and tensors.

Lemma 3.2. *Keep the above notations. Then we have*

$$\begin{aligned} Z(V, \Theta) &\cong \{X \in \Bbbk^{n\times n} \mid (HX)^T = HX\} \\ &= \{X \in \Bbbk^{n\times n} \mid X^T A^{(i_3\cdots i_d)} = A^{(i_3\cdots i_d)} X,\ \forall 1 \le i_3, \cdots, i_d \le n\}. \end{aligned}$$

Remark 3.3. The definition of centers by (V, Θ) is coordinate free, while the other versions by Hessian matrices and slices of tensors are not. Let $P \in \mathrm{GL}(n, \Bbbk)$ be a change of coordinates. If we write

$$Z(A) = \{X \in \Bbbk^{n\times n} \mid X^T A^{(i_3\cdots i_d)} = A^{(i_3\cdots i_d)} X,\ \forall 1 \le i_3, \cdots, i_d \le n\},$$

then $Z(AP^d) = P^{-1} Z(A) P = \{P^{-1} X P \mid X \in Z(A)\}$.

Indeed, assume $P = (p_{ij})$ and denote $B := AP^d$. Notice that

$$B^{(j_3\cdots j_d)} = \left(\sum_{1\le i_1,\cdots,i_d\le n} a_{i_1\cdots i_d} p_{i_1 j_1} p_{i_2 j_2} p_{i_3 j_3} \cdots p_{i_d j_d} \right)_{1\le j_1,j_2\le n}$$

$$= P^T \left(\sum_{1\le i_3,\cdots,i_d\le n} a_{i_1\cdots i_d} p_{i_3j_3}\cdots p_{i_dj_d} \right)_{1\le i_1,i_2\le n} P$$

$$= P^T \left(\sum_{1\le i_3,\cdots,i_d\le n} p_{i_3j_3}\cdots p_{i_dj_d} A^{(i_3\cdots i_d)} \right) P.$$

Denote $C^{(j_3\cdots j_d)} = p_{i_3j_3}\cdots p_{i_dj_d}A^{(i_3\cdots i_d)}$ and $P^{-1} = (q_{ij})$. Then we have

$$A^{(i_3\cdots i_d)} = \sum_{1\le j_3,\cdots,j_d\le n} q_{j_3i_3}\cdots q_{j_di_d} C^{(j_3\cdots j_d)}.$$

Now it follows that

$$\begin{aligned} Z(B) &= \{Y \in \Bbbk^{n\times n} \mid Y^T B^{(j_3\cdots j_d)} = B^{(j_3\cdots j_d)}Y,\ \forall 1 \le j_3,\cdots,j_d \le n\} \\ &= \{Y \in \Bbbk^{n\times n} \mid Y^T P^T A^{(i_3\cdots i_d)}P = P^T A^{(i_3\cdots i_d)}PY,\ \forall 1 \le i_3,\cdots,i_d \le n\} \\ &= \{Y \in \Bbbk^{n\times n} \mid PYP^{-1} \in Z(A)\} = P^{-1}Z(A)P. \end{aligned}$$

Now we recall the definition of direct sum decompositions of forms.

Definition 3.4. A form f is called a direct sum if, after an invertible change of variables, it can be written as a sum of $t \ge 2$ nonzero forms in disjoint sets of variables:

$$f = f_1(x_1,\cdots,x_{a_1}) + \cdots + f_t(x_{a_{t-1}+1},\cdots,x_n).$$

If this is not the case, then f is said to be indecomposable. On the other extreme, f is diagonalizable if the f_i's are forms in only one variable.

If a higher degree form is a direct sum, then there may be various ways of decompositions. However, the decomposition into a direct sum of indecomposable forms is essentially unique thanks to Harrison [6]. It is clear that direct sum decompositions of f correspond to decompositions of its associated symmetric tensor in block terms by d-congruence, and to orthogonal decompositions of its associated symmetric d-linear space.

In this paper we are mainly interested in diagonalizable forms. For this there is no loss of generality in assuming the forms are nondegenerate as in [6], that is no variable can be removed by an invertible linear change of variables. In other words, a form $f \in \Bbbk[x_1,\cdots,x_n]$ is degenerate if there exists an invertible change of variables $x = Py$ such that the resulting form g involves less than n variables. That is, in terms of symmetric multilinear spaces, there exists $0 \ne u \in V$ such that $\Theta(u, v_2,\cdots,v_d) = 0$ for all $v_2,\cdots,v_d \in V$. For the associated symmetric d-tensor A, let A_{i_1} denote the $(d-1)$-tensor $A = (a_{i_1\cdots i_d})_{1\le i_2,\cdots,i_d\le n}$. Then f is degenerate if the A_{i_1}'s are linearly dependent in the space of $(d-1)$-tensors. Similar to the quadratic case, define the radical of the symmetric d-linear space by

$$\operatorname{rad}\Theta := \{u \in V \mid \Theta(u, v_2, \cdots, v_d) = 0, \quad \forall v_2, \cdots, v_d \in V\}$$

and the slicing rank of the symmetric d-tensor by

$$\operatorname{Rank} A := \operatorname{Rank}\{A_1, \cdots, A_n\},$$

namely the size of any maximally linearly independent subset of $\{A_1, \cdots, A_n\}$ viewed as vectors in the space of $(d-1)$-tensors. Note that $\operatorname{Rank} A$ is independent of the ways of slicing along the indices as A is symmetric. We warn the reader that the slicing rank defined here is not the usual tensor rank in the literature, see e.g. [10,18]. The rank of f, denoted by $\operatorname{Rank} f$, is defined to be the essential number r of variables of f, that is, there exists an invertible change of variables $x = Py$ such that the resulting form g involves exactly r variables, say $g = g(y_1, \cdots, y_r)$ and is nondegenerate in $\Bbbk[y_1, \cdots, y_r]$.

Lemma 3.5. *Keep the previous notations. Then*

$$\operatorname{Rank} f = \operatorname{Rank} A = n - \dim \operatorname{rad}\Theta.$$

Remark 3.6. One can easily give a proof mimicking the quadratic situation. As pointed out in [6], any form f can be decomposed as the direct sum of a nondegenerate form g and a zero form h (meaning $h \equiv 0$). The slicing rank is Kruskal's 1-slabs rank [17] for symmetric tensors. The essential number of variables of forms was also considered in [3] via Apolarity Theory and Catalecticant Matrices. An efficient randomized algorithm for computing the essential variables was given by Kayal in [12].

Now we recall some important facts about centers and direct sum decompositions of forms which are useful in this paper. See [6, Propositions 2.3, 4.1 and 4.3] for proofs.

Proposition 3.7. *Suppose $f \in \Bbbk[x_1, \cdots, x_n]$ is a nondegenerate higher degree form. Then*

(1) *The center $Z(f)$ is a commutative subalgebra of $\operatorname{End}(V)$.*
(2) *If $f = f_1 + \cdots + f_t$ is a direct sum decomposition, then $Z(f) \cong Z(f_1) \times \cdots \times Z(f_t)$.*
(3) *If $1 = \epsilon_1 + \cdots + \epsilon_t$ is a decomposition of orthogonal idempotents for the unit of $Z(f)$ (i.e. the identity matrix), then $f = f_1 + \cdots + f_t$ is a direct sum decomposition with the corresponding orthogonal decomposition of symmetric d-linear space $V = \operatorname{Im}\epsilon_1 \oplus \cdots \oplus \operatorname{Im}\epsilon_t$.*
(4) *f is indecomposable over $\Bbbk$ if and only if $Z(f)$ is a local $\Bbbk$-algebra.*
(5) *f is diagonalizable over $\Bbbk$ if and only if $Z(f) \cong \Bbbk \times \cdots \times \Bbbk$ (n copies).*
(6) *The decomposition of f into a direct sum of indecomposable forms is unique up to equivalence and permutation of indecomposable summands.*

(7) *If $K/\Bbbk$ is a field extension, by f_K it is meant treating $f \in K[x_1, \cdots, x_n]$, then $Z(f_K) \cong Z(f) \otimes_{\Bbbk} K$.*

For later applications, we present some more explanations in particular for item (3). Let (V, Θ) be the associated symmetric d-linear space of f under the basis $\alpha_1, \cdots, \alpha_n$. That is,

$$f(x_1, \cdots, x_n) = \Theta\left(\sum_{1 \le i \le n} x_i\alpha_i,\ \ldots,\ \sum_{1 \le i \le n} x_i\alpha_i\right).$$

For simplicity, assume first that $1 = \epsilon_1 + \epsilon_2$ where ϵ_1 and ϵ_2 are a pair of orthogonal idempotents in the center algebra $Z(V, \Theta)$. Then it is obvious that $V = (\epsilon_1 + \epsilon_2)V = \epsilon_1 V \oplus \epsilon_2 V$. For any $v \in V$, let $v^{(i)} = \epsilon_i(v)$ for $i = 1, 2$. Then clearly $v = v^{(1)} + v^{(2)}$. For any $v_1, \cdots, v_d \in V$, we have

$$\begin{aligned}
\Theta(v_1, v_2, \cdots, v_d) &= \Theta(v_1^{(1)} + v_1^{(2)}, v_2^{(1)} + v_2^{(2)}, \cdots, v_d^{(1)} + v_d^{(2)}) \\
&= \sum_{1 \le i_1, i_2, \cdots, i_d \le 2} \Theta(v_1^{(i_1)}, v_2^{(i_2)}, \cdots, v_d^{(i_d)}) \\
&= \Theta(v_1^{(1)}, v_2^{(1)}, \cdots, v_d^{(1)}) + \Theta(v_1^{(2)}, v_2^{(2)}, \cdots, v_d^{(2)}).
\end{aligned}$$

The last equality is a result of the following fact. This is where the orthogonal idempotents play a part.

$$\begin{aligned}
\Theta(v_1^{(i_1)}, v_2^{(i_2)}, \cdots, v_d^{(i_d)}) &= \Theta\left(\epsilon_{i_1}(v_1), \epsilon_{i_2}(v_2), \cdots, \epsilon_{i_d}(v_d)\right) \\
&= \Theta\left(\epsilon_{i_1}\epsilon_{i_2} \cdots \epsilon_{i_d}(v_1), v_2, \cdots, v_d\right) \\
&= 0
\end{aligned}$$

unless $i_1 = i_2 = \cdots = i_d$. Now take a basis $\beta_1, \cdots, \beta_l$ of $\epsilon_1 V$ and a basis $\gamma_1, \cdots, \gamma_m$ of $\epsilon_2 V$. Then we have a new basis $\beta_1, \cdots, \beta_l, \gamma_1, \cdots, \gamma_m$ of V and under it the form becomes

$$\begin{aligned}
&\Theta\left(\sum_{1 \le i \le l} y_i\beta_i + \sum_{1 \le j \le m} z_j\gamma_j, \cdots, \sum_{1 \le i \le l} y_i\beta_i + \sum_{1 \le j \le m} z_j\gamma_j\right) \\
&= \Theta\left(\sum_{1 \le i \le l} y_i\beta_i, \cdots, \sum_{1 \le i \le l} y_i\beta_i\right) + \Theta\left(\sum_{1 \le j \le m} z_j\gamma_j, \cdots, \sum_{1 \le j \le m} z_j\gamma_j\right).
\end{aligned}$$

This is the direct sum decomposition corresponding to the given pair of orthogonal idempotents.

In general, for a given idempotent decomposition $1 = \epsilon_1 + \cdots + \epsilon_t$, assume $\dim \operatorname{Im} \epsilon_i = n_i$ and choose a basis $\beta_{i1}, \cdots, \beta_{in_i}$ for $\operatorname{Im} \epsilon_i$. Then $\beta_{11}, \cdots, \beta_{1n_1}, \cdots, \beta_{t1}, \cdots, \beta_{tn_t}$ is a basis of V and there exists a $P = (p_{ij}) \in \mathrm{GL}(n, \Bbbk)$ such that

$$(\alpha_1, \cdots, \alpha_n) = (\beta_{11}, \cdots, \beta_{1n_1}, \cdots, \beta_{t1}, \cdots, \beta_{tn_t})P.$$

For convenience, simplify the indices of the new basis as $\beta_1, \cdots, \beta_n$. Then we have

$$\begin{aligned} f(x_1, \cdots, x_n) &= \Theta\left(\sum_{1\le i\le n} x_i\alpha_i,\ \ldots,\ \sum_{1\le i\le n} x_i\alpha_i\right) \\ &= \Theta\left(\sum_{1\le j\le n}\sum_{1\le i\le n} p_{ji}x_i\beta_j,\ \ldots,\ \sum_{1\le j\le n}\sum_{1\le i\le n} p_{ji}x_i\beta_j\right) \\ &= \Theta\left(\sum_{j=1}^{n_1}\sum_{1\le i\le n} p_{ji}x_i\beta_j,\ \ldots,\ \sum_{j=1}^{n_1}\sum_{1\le i\le n} p_{ji}x_i\beta_j\right) \\ &\quad + \cdots + \\ &\quad \Theta\left(\sum_{j=n-n_t+1}^{n}\sum_{1\le i\le n} p_{ji}x_i\beta_j,\ \ldots,\ \sum_{j=n-n_t+1}^{n}\sum_{1\le i\le n} p_{ji}x_i\beta_j\right). \end{aligned}$$

Note that the last equality is due to the orthogonality of the ϵ_i's. Let $y = Px$ be the invertible change of variables and denote

$$\begin{aligned} g_1(y_1, \cdots, y_{n_1}) &= \Theta\left(\sum_{j=1}^{n_1} y_j\beta_j,\ \ldots,\ \sum_{j=1}^{n_1} y_j\beta_j\right), \\ &\ \ \vdots \\ g_t(y_{n-n_t+1}, \cdots, y_n) &= \Theta\left(\sum_{j=n-n_t+1}^{n} y_j\beta_j,\ \ldots,\ \sum_{j=n-n_t+1}^{n} y_j\beta_j\right). \end{aligned}$$

Now we have the desired direct sum decomposition

$$f = g_1(y_1, \cdots, y_{n_1}) + \cdots + g_t(y_{n-n_t+1}, \cdots, y_n).$$

4. Main results

In this section, take the ground field $\Bbbk = \mathbb{C}$, the case of our interest. In order to obtain criteria and algorithms for diagonalizable forms we put their algebro-geometric properties, in particular the obvious smoothness, into consideration.

Recall that a form $f \in \mathbb{C}[x_1, \cdots, x_n]$ is called smooth, if the simultaneous equations

$$\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

have no nonzero solutions, see e.g. [8]. In terms of the associated symmetric multilinear space (V, Θ), this is equivalent to

$$\Theta(u, \cdots, u, v_d) = 0, \ \forall v_d \in V \ \Rightarrow \ u = 0.$$

It is clear that, if f is smooth, then it is nondegenerate.

It was observed by Harrison [6] that the smoothness of f implies that its center $Z(f)$ has no nontrivial nilpotent elements. In the present situation, we have

Lemma 4.1. *Suppose $f \in \mathbb{C}[x_1, \cdots, x_n]$ is a smooth form of degree d. Then $Z(f) \cong \mathbb{C} \times \cdots \times \mathbb{C}$, and $\dim Z(f)$ is equal to the number of indecomposable summands of f and $\dim Z(f) \leq n$. In particular, f is diagonalizable if and only if $\dim Z(f) = n$.*

Proof. For the sake of completeness, we recall Harrison's argument that $Z(f)$ has no nontrivial nilpotent elements if f is smooth. Assume the contrary, there is a nilpotent element $\phi \in Z(f)$ with $\phi^{m+1} = 0$ while $\phi^m \neq 0$ for some $m \geq 1$. Then there is some $v \in V$ such that $\phi^m(v) \neq 0$. Hence

$$\begin{aligned}\Theta(\phi^m(v), \cdots, \phi^m(v), v_d) &= \Theta(\phi^{2m}(v), \phi^m(v), \cdots, \phi^m(v), v, v_d) = \cdots \\ &= \Theta(\phi^{(d-1)m}(v), v, \cdots, v, v_d) = 0\end{aligned}$$

for all $v_d \in V$ as $(d-1)m \geq m+1$. Now the smoothness condition of f forces $\phi^m(v) = 0$. This is absurd.

The rest is easy. Since $Z(f)$ is commutative with zero radical, then by the Wedderburn-Artin theorem of semisimple algebras, $Z(f) \cong \mathbb{C} \times \cdots \times \mathbb{C}$. Now by Proposition 3.7, indecomposable direct summands of f are in bijection with orthogonal primitive idempotents of $Z(f)$. Hence the number of indecomposable direct summands of f is exactly $\dim Z(f)$. As is obvious that this number can not exceed the number of variables of f, hence $\dim Z(f) \leq n$.

Finally, if f is diagonalizable, then its number of indecomposable summands is n, thus $\dim Z(f) = n$. Conversely, if $\dim Z(f) = n$, then $Z(f) \cong \mathbb{C} \times \cdots \times \mathbb{C}$ (n copies) since $Z(f)$ is a semisimple commutative $\mathbb{C}$-algebra. It follows by item (5) of Proposition 3.7 that f is diagonalizable. □

Remark 4.2. As indicated in [6], one may define centers for nondegenerate quadratic forms which are essentially symmetric matrices. The diagonal idempotent matrices are obviously symmetric, hence in the center, and through symmetric bilinear forms they provide a diagonal decomposition for quadratic forms. The present lemma may be seen as a generalization of this fact to higher degree forms, but with the great distinction that the existence of nontrivial idempotents is not automatic (in fact, almost impossible according to Proposition 4.3 in below).

From now on, let $\mathrm{V}_{n,d} \subset \mathbb{C}[x_1, \cdots, x_n]$ denote the linear space of forms of degree d in n variables. Let $\mathrm{U}_{n,d} \subset \mathrm{V}_{n,d}$ be the set of smooth forms. It is well known that $\mathrm{U}_{n,d}$ is an open subset of $\mathrm{V}_{n,d}$ defined by an irreducible polynomial $\Delta_{n,d} \neq 0$, where $\Delta_{n,d}$ is the discriminant (see Chapter 13 in [5]). Let $\mathrm{Diag}_{n,d} \subset \mathrm{U}_{n,d}$ denote the subset of diagonalizable forms.

In the following we show that diagonalizable higher degree forms should be very special and rare.

Proposition 4.3. $\mathrm{Diag}_{n,d}$ *is a proper closed subset of* $\mathrm{U}_{n,d}$.

Proof. Obviously, $\mathrm{Diag}_{n,d}$ is not empty. Thanks to [22, Corollary 6.1], $\mathrm{Diag}_{n,d}$ is a proper subset of $\mathrm{U}_{n,d}$. Note by Lemma 3.2 that the center $Z(f)$ of a form f is the solution space of a system of linear equations, written simply as $Cy = 0$, which is obtained by combining together all the matrix equations therein. Consider the entries of C as linear forms of the coefficients of a general form in $\mathrm{U}_{n,d}$. Due to Lemma 4.1, a form $f \in \mathrm{U}_{n,d}$ is diagonalizable if and only if $\dim Z(f) = n$, thus if and only if $\mathrm{Rank}\, C = n^2 - n$. Again by Lemma 4.1, we have $\mathrm{Rank}\, C \geq n^2 - n$ since $n^2 - \mathrm{Rank}\, C = \dim Z(f) \leq n$. Now it follows that $\mathrm{Diag}_{n,d}$ is a closed subset of $\mathrm{U}_{n,d}$ defined by all the $(n^2 - n + 1)$-minors of C. □

In summary, we have the following criteria for diagonalizable forms of higher degree. For convenience, we consider $Z(f) \subset \mathbb{C}^{n\times n}$ and let D denote the subalgebra of $\mathbb{C}^{n\times n}$ consisting of all diagonal matrices.

Theorem 4.4. *Suppose* $f \in \mathrm{V}_{n,d}$ *is nondegenerate. Then the following statements are equivalent:*

(1) *The form* f *is diagonalizable.*
(2) *The center* $Z(f) \cong D$.
(3) f *is smooth and* $\dim Z(f) = n$.
(4) $Z(f)$ *is semisimple and* $\dim Z(f) = n$.
(5) $\dim Z(f) = n$ *and any basis of* $Z(f)$ *consists of diagonalizable matrices.*
(6) $\dim Z(f) = n$ *and* $Z(f)$ *has a basis consisting of rank 1 and trace 1 matrices.*

Proof. The equivalence of (1), (2), (3) and (4) is already contained in Lemma 4.1. Clearly, (2) implies (5) and (6). Assume (5), then by a simultaneous diagonalization of any basis (since $Z(f)$ is commutative) one easily obtains $Z(f) \cong D$. Finally assume (6). Note that a rank 1 and trace 1 matrix is obviously idempotent. The condition of (6) says exactly that $Z(f)$ has a basis consisting of n commuting idempotent matrices. This immediately implies $Z(f) \cong D$. □

Now we turn to the problem of orthogonal or unitary diagonalizations of forms and symmetric tensors. This has been intensively studied in e.g. [19,1,13] and called symmetrically odeco (over $\mathbb{R}$) or udeco (over $\mathbb{C}$) therein. Apparently orthogonally or unitarily diagonalizable forms are *a priori* diagonalizable. On the other hand, notice that the diagonalization (if exists) of a form is essentially unique due to the seminal work of Harrison [6]. So the key is the diagonalization, and the orthogonality or unitarity is a property which can be checked directly. Therefore, it is very easy to decide whether or not a higher degree form or symmetric tensor is orthogonally or unitarily diagonalizable based upon the previous theorem.

Corollary 4.5. *Suppose $f \in \mathrm{Diag}_{n,d}$ and $P = (p_{ij})_{1\le i,j\le n}$ is an invertible matrix such that*

$$f = \sum_{1\le i\le n} \alpha_i \left(\sum_{1\le j\le n} p_{ij}x_j \right)^d .$$

Then f is orthogonally (resp. unitarily) diagonalizable if and only if, after appropriate scaling of each row, P is orthogonal (resp. unitary).

Proof. Assume f is orthogonally (resp. unitarily) diagonalizable, that is there exist an orthogonal (resp. unitary) matrix $Q = (q_{ij})_{1\le i,j\le n}$ such that

$$f = \sum_{1\le i\le n} \lambda_i \left(\sum_{1\le j\le n} q_{ij}x_j \right)^d .$$

By item (6) of Proposition 3.7, there exists $\tau_i \in \mathbb{C}^*$ for all $1 \le i \le n$ such that

$$\sum_{1\le j\le n} q_{\sigma(i)j}x_j = \tau_i \sum_{1\le j\le n} p_{ij}x_j,$$

where σ is a permutation of $\{1,\dots,n\}$. That is, P is orthogonal (resp. unitary) up to scaling by a suitable diagonal matrix. The converse is trivial. □

As an example of our approach to symmetrically odeco and udeco, we give a simple proof for a main result in [1,13]: the set of symmetrically odeco tensors can be described by equations of degree 2. We will adopt the version of characterization for degree 3 in [13, Theorems 3.3 and 3.6], but prove the result for all degrees. In the following we use the notations of Lemma 3.2 and we stick to the notions in [1,13] for coherence.

Proposition 4.6. *A nondegenerate real (resp. complex) symmetric tensor* $A = (a_{i_1\cdots i_d})_{1\le i_1,\cdots,i_d\le n}$ *is odeco if and only if the* $A^{(i_3\cdots i_d)}$*'s pairwise commute (resp. and are diagonalizable).*

Proof. We only prove the real case, as the complex case is similar.

Assume A is odeco. Then there exists an orthogonal matrix O such that AO^d is diagonal. By Lemma 3.2, it is immediate that $Z(AO^d) = \mathrm{D}_n(\mathbb{R})$, the subalgebra of $\mathbb{R}^{n\times n}$ consisting of all diagonal matrices. It follows by Remark 3.3 that $Z(A) = O\,\mathrm{D}_n(\mathbb{R})O^T$. Again by Lemma 3.2, $O^TA^{(i_3\cdots i_d)}O$ commutes with $\mathrm{D}_n(\mathbb{R})$ for all $1 \le i_3,\cdots,i_d \le n$. Thus $O^TA^{(i_3\cdots i_d)}O$ is diagonal for all $1 \le i_3,\cdots,i_d \le n$. It follows right away that the $A^{(i_3\cdots i_d)}$'s pairwise commute.

Conversely, assume the $A^{(i_3\cdots i_d)}$'s pairwise commute. As they are all symmetric, there is an orthogonal matrix O such that $O^TA^{(i_3\cdots i_d)}O$ is diagonal for all $1 \le i_3,\cdots,i_d \le n$. As $\mathrm{Rank}\,A = n$, the linear span of all the $O^TA^{(i_3\cdots i_d)}O$ is $\mathrm{D}_n(\mathbb{R})$. Then by Lemma 3.2, we have $Z(A) = O\,\mathrm{D}_n(\mathbb{R})O^T$. Now It follows by items (3) and (5) of Proposition 3.7 that A is odeco. □

Remark 4.7. The previous simple criteria provide several ways to detect and determine the (orthogonal, or unitary) diagonalizations of higher degree forms. For example, by (3) of Theorem 4.4 compute $\dim Z(f)$ (via the rank of the matrix C in the proof of Proposition 4.3) and detect the smoothness of f (e.g., by the Jacobian criterion); by (4), compute $Z(f)$ and detect the semisimplicity of a basis (via the minimal polynomial of any basis element, using Euclid algorithm to detect whether the polynomial has multiple roots); by (5), compute $Z(f)$ and simultaneously diagonalize a basis; by (6), compute $Z(f)$ and find a complete set of orthogonal primitive idempotents from any chosen basis of $Z(f)$.

In the following we will focus on the fourth and give in detail a theoretical algorithm later on. The reason is two-fold: on the one hand, it detects and determines the explicit diagonalization simultaneously; on the other hand, it is relatively simple involving only linear and quadratic equations.

Algorithm 4.8. Take any $f \in \mathrm{V}_{n,d}$. Let A be the associated symmetric d-tensor.

Step 1: Detect the nondegeracy. Compute $\mathrm{Rank}\,A$ according to Lemma 3.5. If $\mathrm{Rank}\,A = n$, then continue; otherwise, say $\mathrm{Rank}\,A = r < n$, reduce the form f into a nondegenerate one and restart in $\mathrm{V}_{r,d}$.
Step 2: Compute the center. Solve the linear equations as in Lemma 3.2. If $\dim Z(f) < n$, then f is not diagonalizable and we stop; otherwise, choose a basis $P_1,\dots,P_n$ of $Z(f)$.
Step 3: Compute the idempotents. Let $\lambda_1,\dots,\lambda_n$ be indeterminates. Consider the matrix $\epsilon = \sum_{1\le i\le n}\lambda_iP_i$ and impose the conditions: $\mathrm{Rank}\,\epsilon = 1$ (i.e., all the 2×2-

minors are zero) and $\mathrm{trace}(\epsilon) = 1$. Solve the equations. If there exists no solutions, then f is not diagonalizable and we stop; otherwise choose an ϵ and continue.

Step 4: Diagonalize the form. Decompose f according to the pair $(\epsilon, 1-\epsilon)$ of orthogonal idempotents as in the explanation after Proposition 3.7. Then we have $f = l_1(x_1, \ldots, x_n)^d + g$ where g is a nondegenerate form of degree d in $n-1$ variables which are linear forms of $x_1, \ldots, x_n$ independent from l_1. Then return to step 1 and replace f by $g \in \mathrm{V}_{n-1,d}$. If the input f is diagonalizable, then in $n-1$ steps the procedure stops and we end up with a diagonalization of f.

Step 5: Check orthogonality or unitarity. By direct computation, verify whether the resulting change of variables is orthogonal, or unitary, or neither.

Remark 4.9. In principle, our algorithm can be easily adapted to diagonalize higher degree forms over non algebraically closed fields. That is, one can work firstly on the algebraic closure of the ground field and check directly if the final diagonalization can be realized on the given ground field. This is guaranteed by Harrison's uniqueness result of decompositions of higher degree forms. In comparison with some previous works [1,4,11,13,14,19], our algorithm seems simple at first sight. In order to simplify the quadratic equations of the previous Step 3, an obvious reduction is to diagonalize the P_i's simultaneously (as they are commuting). This is more or less equivalent to the tasks of computing eigenvalues and eigenvectors for which there are sophisticated algorithms.

5. Some examples

In this section, we provide some examples which are solvable by hand. The first two examples are the well known canonical forms of binary quartics and ternary cubics. The last two examples are typical ternary cubics and quaternary quartics involving all possible monomials. These will help to fully elucidate our criteria and algorithms of diagonalization. There are two main steps in the computations. The first is a routine calculation of linear equations according to Lemma 3.2 which gives centers as linear spaces. The second is to determine the algebraic structure of centers and single out idempotents to decompose the forms according to Steps 3 and 4 of Algorithm 4.8. Along the way, one can also see clearly how to decompose forms over non algebraically closed fields.

Example 5.1 *(Binary quartics).* Let $f_t = x_1^4 + x_2^4 + tx_1^2x_2^2 \in \Bbbk[x_1, x_2]$. Then it is easy to see that f_t is smooth if and only if $t \neq \pm 2$. Moreover, if $t = 0$, then clearly f_t is diagonal. In the following suppose $t \neq 0, \pm 2$. The Hessian matrix of f_t is $\begin{pmatrix} 12x_1^2 + 2tx_2^2 & 4tx_1x_2 \\ 4tx_1x_2 & 12x_2^2 + 2tx_1^2 \end{pmatrix}$. Then by Lemma 3.2, we have $Z(f_t) \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} | a \in \Bbbk \right\} \cong \Bbbk$ if $t \neq \pm 6$, $Z(f_6) \cong \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} | a, b \in \Bbbk \right\} \cong \Bbbk \times \Bbbk$, and

$Z(f_{-6}) \cong \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} | a, b \in \Bbbk \right\} \cong \Bbbk[x]/(x^2+1)$. Therefore, if $t \neq \pm 6$, then f_t is absolutely indecomposable (namely, it remains indecomposable over any extension of the ground field).

For $t = 6$, we have the following decomposition of the unit into a sum of orthogonal idempotents:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Note that $(1,1)$ (resp. $(1,-1)$) is a basis of the image of the first (resp. second) idempotent. Then $(1,1)$ and $(1,-1)$ form a basis of $\Bbbk^2$ under which f_6 is diagonalizable: $f_6 = \frac{1}{2}[(x_1+x_2)^4 + (x_1-x_2)^4]$.

For $t = -6$, if $\sqrt{-1} \in \Bbbk$, similarly we have the following decomposition of the unit:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix},$$

and $(1, \sqrt{-1})$ (resp. $(1, -\sqrt{-1})$) is a basis of the image of the first (resp. second) idempotent. Then under the new basis of $\Bbbk^2$ the form f_{-6} is diagonalizable: $f_{-6} = \frac{1}{2}[(x_1+\sqrt{-1}x_2)^4 + (x_1-\sqrt{-1}x_2)^4]$. If $\sqrt{-1} \notin \Bbbk$, then $\Bbbk[x]/(x^2+1)$ is a field and thus f_{-6} is indecomposable (but not absolutely). In addition, if $\sqrt{2} \in \Bbbk$ as well, then f_6 is orthogonally diagonalizable and f_{-6} is unitarily diagonalizable.

Example 5.2 *(Ternary cubics).* Consider the normal form of nonsingular ternary cubics $f_\lambda = x_1^3 + x_2^3 + x_3^3 + 6\lambda x_1x_2x_3$. Clearly f_0 is already diagonal. In the following suppose $\lambda \neq 0$. By a straightforward computation we have $Z(f_\lambda) \cong \Bbbk$ if $\lambda^3 \neq 1$, thus f_λ is absolutely indecomposable in this case. If $\lambda^3 = 1$ but $\lambda \neq 1$, then $Z(f_\lambda) \cong \Bbbk \times \Bbbk \times \Bbbk$, consequently $f_\lambda = \frac{1}{3}[(\lambda x_1+x_2+x_3)^3 + (x_1+\lambda x_2+x_3)^3 + (x_1+x_2+\lambda x_3)^3]$ is diagonalizable (but neither orthogonal nor unitary). If $\lambda = 1$, then $Z(f_1) \cong \Bbbk \times \Bbbk[x]/(x^2+x+1)$. In this case, according to the explanation after Proposition 3.7 take the change of variables $x_1 = y_1 + y_2,\ x_2 = y_1 + y_3,\ x_3 = y_1 - y_2 - y_3$ and we have the corresponding direct sum decomposition $f_1 = 9y_1^3 - 9(y_2^2y_3 + y_2y_3^2)$ where the center of $y_2^2y_3 + y_2y_3^2$ is isomorphic to $\Bbbk[x]/(x^2+x+1)$. If $\Bbbk$ does not contain a primitive cubic root of unity, then $\Bbbk[x]/(x^2+x+1)$ is a field and thus $y_2^2y_3 + y_2y_3^2$ is indecomposable by item (4) of Proposition 3.7. If $\omega \in \Bbbk$ is a primitive cubic root of unity, then $\Bbbk[x]/(x^2+x+1) \cong \Bbbk \times \Bbbk$ and we can further decompose $y_2^2y_3 + y_2y_3^2$. In this case we have the diagonalization of f_1 as

$$\frac{1}{3}[(x_1+x_2+x_3)^3 + (x_1+\omega x_2+\omega^2 x_3)^3 + (x_1+\omega^2 x_2+\omega x_3)^3].$$

Furthermore, if $\Bbbk$ also contains $\sqrt{3}$, then f_1 is unitarily diagonalizable.

Example 5.3. Consider the cubic form

$$f = x_1^3 - 3x_1^2x_2 + 3x_1x_2^2 + 3x_1^2x_3 + 3x_1x_3^2 - 6x_1x_2x_3 + 13x_2^3 - 3x_2^2x_3 - 9x_2x_3^2 + 15x_3^3$$

in $\mathbb{Q}[x_1,\ x_2,\ x_3]$. In accordance with Lemma 3.2, let

$$A^{(1)} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 13 & -1 \\ -1 & -1 & -3 \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & -3 \\ 1 & -3 & 15 \end{pmatrix}.$$

Then by a direct calculation, $Z(f) = \{X \in \mathbb{Q}^{3\times 3} \mid A^{(i)}X = X^TA^{(i)},\ 1 \le i \le 3\} = \bigoplus_{1\le i\le 3} \mathbb{Q}X^{(i)}$, where

$$X^{(1)} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X^{(2)} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X^{(3)} = \begin{pmatrix} 0 & 1 & -5 \\ 0 & 0 & 1 \\ 0 & -1 & 6 \end{pmatrix}.$$

Note that $X^{(1)}$ is of rank 1 and trace 1, hence it is a primitive idempotent. Then according to the pair $(X^{(1)}, I_3 - X^{(1)})$ of idempotents we have the following decomposition

$$(x_1 - x_2 + x_3)^3 + (14x_2^3 - 6x_2^2x_3 - 6x_2x_3^2 + 14x_3^3).$$

Apply the same process to $g := 14x_2^3 - 6x_2^2x_3 - 6x_2x_3^2 + 14x_3^3$. In fact, we can read from the $X^{(i)}$'s that $Z(g) = \mathbb{Q}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{Q}\begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix} \cong \mathbb{Q}[\sqrt{2}]$. This is a quadratic extension of $\mathbb{Q}$. It follows immediately that g is indecomposable over $\mathbb{Q}$. Consequently $Z(f) \cong \mathbb{Q} \times \mathbb{Q}[\sqrt{2}]$ and f is a direct sum but not diagonalizable over $\mathbb{Q}$.

However, over any field extension $K/\mathbb{Q}$ with $\sqrt{2} \in K$, one has easily $Z(f) \otimes_{\mathbb{Q}} K \cong K \times K \times K$ and f is diagonalizable over K:

$$(x_1 - x_2 + x_3)^3 + [(1+\sqrt{2})x_2 + (1-\sqrt{2})x_3)]^3 + [(1-\sqrt{2})x_2 + (1+\sqrt{2})x_3]^3.$$

Finally by Corollary 4.5, it is clear that any diagonalization of f is neither orthogonal nor unitary.

Example 5.4. Consider the following rational quartic form

$$\begin{aligned} f =& x_1^4 + 4x_1^3x_2 + 4x_1^3x_3 + 4x_1^3x_4 - 12x_1^2x_2^2 + 12x_1^2x_2x_3 - 24x_1^2x_2x_4 - 12x_1^2x_3^2 \\ &- 24x_1^2x_3x_4 + 24x_1^2x_4^2 + 4x_1x_2^3 - 24x_1x_2^2x_3 + 12x_1x_2^2x_4 - 24x_1x_2x_3^2 + 96x_1x_2x_3x_4 \\ &- 24x_1x_2x_4^2 + 4x_1x_3^3 + 12x_1x_3^2x_4 - 24x_1x_3x_4^2 + 4x_1x_4^3 + x_2^4 + 4x_2^3x_3 + 4x_2^3x_4 \\ &+ 24x_2^2x_3^2 - 24x_2^2x_3x_4 - 12x_2^2x_4^2 + 4x_2x_3^3 - 24x_2x_3^2x_4 + 12x_2x_3x_4^2 + 4x_2x_4^3 + x_3^4 \\ &+ 4x_3^3x_4 - 12x_3^2x_4^2 + 4x_3x_4^3 + x_4^4. \end{aligned}$$

Similar to the previous example, let A denote the associated symmetric tensor of f and set $A^{(i_3 i_4)}$ as in Lemma 3.2 to compute the center to get $Z(f) = \bigoplus_{1\le i\le 4} \mathbb{Q}X^{(i)}$, where

$$X^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

$$X^{(3)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad X^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Denote $Y = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ and notice that $X^{(2)} = I_2 \otimes Y,\ X^{(3)} = Y \otimes I_2$ and $X^{(4)} = Y \otimes Y$. Now it is ready to see that

$$Z(f) \cong \mathbb{Q}[x]/(x^2 - x + 1) \otimes \mathbb{Q}[x]/(x^2 - x + 1) \cong \mathbb{Q}[\sqrt{-3}] \times \mathbb{Q}[\sqrt{-3}].$$

So f is not diagonalizable over $\mathbb{Q}$ by Theorem 4.4, but is a direct sum of two rational indecomposable forms by item (3) of Proposition 3.7. In light of the structure of $Z(f)$, it is not hard to find out a pair of orthogonal idempotents:

$$\epsilon_1 = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 & 2 \\ 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix}, \quad \epsilon_2 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & -2 \\ -1 & 2 & 2 & -1 \\ -1 & 2 & 2 & -1 \\ -2 & 1 & 1 & 1 \end{pmatrix}.$$

Accordingly we take the following change of variables

$$x_1 = \frac{1}{3}(y_1 + y_2 + y_3), \quad x_2 = \frac{1}{3}(y_1 + y_3 + y_4),$$
$$x_3 = \frac{1}{3}(y_2 + y_3 + y_4), \quad x_4 = \frac{1}{3}(y_1 + y_2 + y_4).$$

The corresponding direct sum decomposition is $f = t(y_1, y_2) + t(y_3, y_4)$ with

$$t(\alpha, \beta) = \frac{1}{9}(-\alpha^4 + 4\alpha^3\beta + 12\alpha^2\beta^2 + 4\alpha\beta^3 - \beta^4).$$

Furthermore over any field extension $K/\mathbb{Q}$ with $\sqrt{-3} \in K$, the center $Z(f) \otimes_{\mathbb{Q}} K \cong K \times K \times K \times K$ and f can be diagonalized as:

$$(x_1 + \omega x_2 + \omega^2 x_3 + x_4)^4 + (\omega x_1 + x_2 + x_3 + \omega^2 x_4)^4 + (\omega^2 x_1 + x_2 + x_3 + \omega x_4)^4$$
$$+ (x_1 + \omega^2 x_2 + \omega x_3 + x_4)^4$$

where $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. Therefore any diagonalization of f is neither orthogonal nor unitary.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The authors would like to thank Professor Minghua Lin for a careful reading through the manuscript and for providing many helpful suggestions. They would also like to thank the referees for many useful comments and advice which help to improve the exposition tremendously.

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